Modelling Stock Prices

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1 Introduction

This post will introduce some of the basic underlying principles behind modelling stock prices. It will provide the reader with a solid background to the key mathematical concepts used. Following this, the post will then use an example of a stock listed on the FTSE 100 index and demonstrate how the mathematical model can be implemented. The stock that was chosen here is Diageo plc (DGE).

2 Brownian Motion

Geometric Brownian Motion is a continuous time stochastic process in which a randomly varying quantity follows a Brownian Motion with a drift (Ross, 2014). A particularly useful application is the modelling of finance. In particular modelling a future stock price movement as following geometric Brownian motion.

(i) A random variable $Z \sim N(0,1)$ if it's density function is given by $\phi = \frac{e^{-z^2/2}}{\sqrt{2\pi}}$. The function $\Phi(z) = \int_{-\infty}^{z} \phi(u)du$ denotes the distribution function of a standard normal variable. More generally, a random variable V has a normal distribution with mean μ and standard deviation $\sigma > 0$ this is expressed as $V \sim N(\mu, \sigma^2)$. A special case of this is where a random variable $X = exp(\sigma Z + \mu)$ where $Z \sim N(0,1)$, is log-normal (μ, σ^2). An example of where this is used is modelling stock prices that are unable to be negative as they are bounded by zero. Note that the parameters μ and σ here are the mean and standard deviation of $logX$.(Sharpe, 2004)

The stochastic differential equation is as follows:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t. \tag{1}$$

Let S_0 denote the price of a stock at time t = 0 at regular intervals of t = 1, t = 2, to t = n. S_t is therefore the price of the stock at time t. μ_t is the drift coefficient (mean) denoted as a percentage and is the rate of change in the conditional mean for a random stochastic process. (Sharpe, 2004) within a financial context drift is the future expectations to a shares price.

The other coefficient from equation 1 is σ_t and is the volatility coefficient (standard deviation) and is the rate at which something varies. It is the magnitude of uncertainty (Sharpe, 2004). Within a financial model, this is the rate of change in a share price and allows us to gain an insight into the stability of the price of a share.

In equation 1 both coefficients are functions of time too.

(ii) Given that $\sigma_t = \sigma$ and $\mu_t = \mu$ equation 1 can be rewritten as:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{2}$$

the left hand side of 2 is similar to the derivative of $ln(S_t)$ then:

$$\frac{d}{S_t} ln(S_t) = \frac{1}{S_t}. \tag{3}$$

This can be rearranged to give:

$$dln(S_t) = \frac{dS_t}{S_t}. \tag{4}$$

Consequently, equation 2 becomes

$$dln(S_t) = \mu dt + \sigma dW_t. \tag{5}$$

Given that $dS_t = (\mu dt + \sigma dW_t)S_t$ then $(dS_t)^2$ can be rewritten as:

$$(dS_t)^2 = S_t^2[(\mu dt)^2 + (\sigma dW_t)^2 + 2\mu\sigma dt dW_t], \tag{6}$$

imposing conditions $(dt)^2 = 0$, $(dW_t)^2 = dt$ and $dtdW_t = 0$ it reduces to:

$$(dS_t)^2 = \sigma^2 S_t^2 dt, \tag{7}$$

given that an approximation of a function evaluated about zero, written as a power series for a two variable case can be expressed as:

$$\begin{aligned} f(a,b) = f(0,0) + f_a(0,0) + f_a(0,0)a + f_b(0,0)b + \frac{f_{aa}(0,0)}{2!}a^2 \\ + \frac{f_{bb}(0,0)}{2!}b^2 + f_{ab}(0,0)ab + O, \end{aligned} \tag{8}$$

Itô's lemma (Geo) can be written as:

$$df = \frac{\partial f}{\partial t}dt + f' dS_t + \frac{1}{2}f''(dS_t)^2. \tag{9}$$

When $f = ln(S_t)$ then (9) can be applied and substituting gives:

$$dln(S_t) = \frac{\partial ln(S_t)}{\partial t}dt + \frac{1}{S_t}S_t[\mu dt + \sigma dW_t] + \frac{1}{2}(-\frac{1}{S_t}^2)\sigma^2 S_t^2 dt, \quad (10)$$

simplifying gives:

$$dln(S_t) = 0 + \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt, \quad (11)$$

factorising:

$$dln(S_t) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t, \quad (12)$$

using $dln(S_t) = ln(S_t) - ln(S_0)$

$$ln\left(\frac{S_t}{S_0}\right) = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \quad (13)$$

Since (13) is simply shorthand for an integral formula, it can be integrated and written as:

$$ln\left(\frac{S_t}{S_0}\right) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t, \quad (14)$$

taking the exponential of each side gives:

$$exp\left(ln\left(\frac{S_t}{S_0}\right)\right) = exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right], \quad (15)$$

the solution for S_t becomes:

$$S_t = S_0 exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right], \quad (16)$$

where at $t = 0$ then $S_0 \neq 0$, $W_0 \neq 0$ and S_0 is a constant.

Assumptions:

- Stock prices are continuous in both time and value.
- Only the present value of the stock is relevant to predicting the future price, this is known as a Markov process and is a particular type of stochastic process.
- Proportional returns on the underlying stock over short time periods is normally distributed.
- The return for a continuously compounded stock is normally distributed.
- The stock does not pay dividends.

(Marathe and Ryan, 2005b)

3 Diageo plc, Ticker: DGE

Selecting Diageo plc from the FTSE 100 index on the stock market it can now be demonstrated how Brownian motion works. To do this the closing share price data for Diageo plc is collected for each trading data for the three consecutive months from 1 August 2019 to 31 October 2019 inclusive.

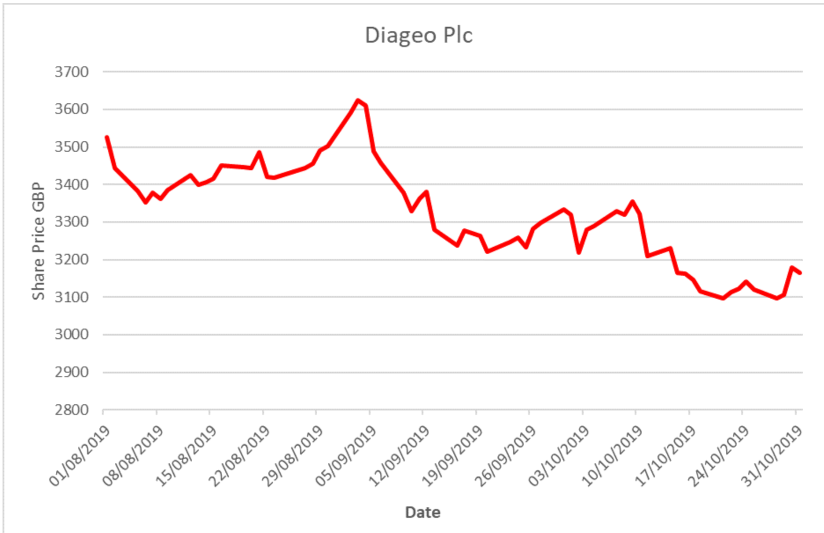

Figure 1: Graph of stock price for Diagio plc between 1 August 2019 and 31 October 2019.

(i) The condition that share prices much have to satisfy in order to be classified as geometric Brownian motion are, that the shares must follow a log-normal distribution $GBM \sim (\mu_y, \sigma_t^2)$. (Marathe and Ryan, 2005a) The test here is that:

H_0: The daily return follows a log-normal distribution.
H_1: The daily return does not follow a log-normal distribution.

The daily return is calculated by taking the current day over the previous day and taking a log of this. This is expressed mathematically as $u = ln(\frac{S_t}{S_{t-1}})$. Figure 2 is a quantile-quantile (q-q) plot and is an exploratory graphical device used to check the validity of a distributional assumption for a data set. The q-q plot provides a visual comparison of the sample quantiles to the corresponding theoretical quantities. In general, if the points in a q-q plot depart from a straight line, then the assumed distribution is called into question.(Scott) Consequently, here the data points to not drastically depart from a straight line and can be considered to follow a log-normal distribution. A more rigorous approach is the Shapiro Wilks test (Marathe and Ryan, 2005a) the output for this test are given in Figure 3 this illustrates that the p-value = $0.2065 > 0.05$, this means that H_0 is retained at the 5% level. Consequently, it can be

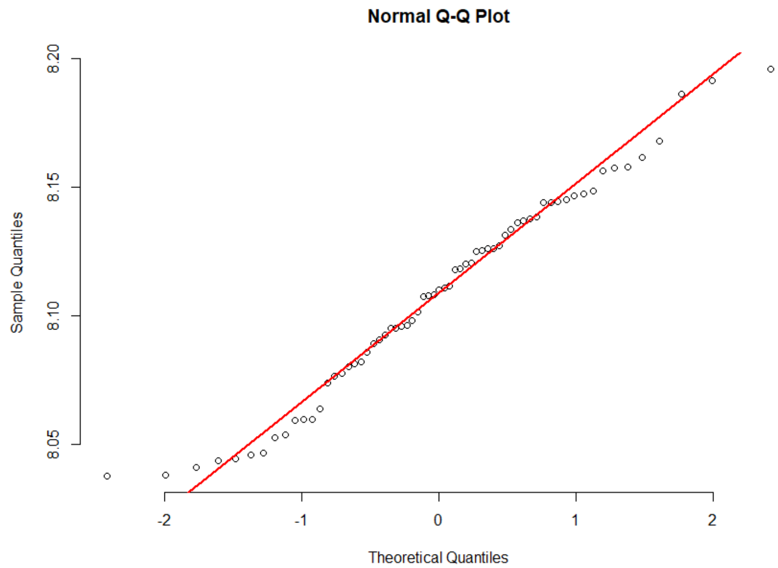

Figure 2: Plot for Normality

concluded that the daily return follows a log-normal distribution and can therefore be classified as geometric Brownian motion.

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        Shapiro-Wilk normality test

data:  Log_Close$Log_Close
W = 0.97482, p-value = 0.2065
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Figure 3: Statistical Output of the Shapiro Wilks Test

(ii)

Evaluation of Volatility σ

S_t denotes share closing price at end of t-th trading period with τ as the length of time interval between two consecutive trading periods expressed in years ($\tau = t_i - t_{i-1}$). If u_t is the length of the daily return over interval τ

$$u_t = ln\left(\frac{S_t}{S_{t-1}}\right) \; for \; t = 1...n \tag{17}$$

The the unbiased estimator $\bar{u}$ of the log of the return is:

$$\bar{u} = \frac{1}{n}\sum_{t=1}^{n} u_t \tag{18}$$

This gives standard deviation:

$$v = \sqrt{\frac{1}{n-1}\sum_{t=1}^{n}(u_t - \bar{u})^2} \quad (19)$$

To evaluate σ, the following formula can be used(Dmouj, 2006)

$$\sigma = \frac{v}{\sqrt{\tau}} \quad (20)$$

Inputting the closing share prices for Diageo plc from 1 August 2019 to 31 October 2019 obtained from Yahoo Finance, gives a value of n = 64. Using equation 18 gives $\bar{u}$ = -0.00196 and subsequently, using equation 19 gives v = 0.013155. The time interval τ is $\frac{1}{252}$. Inputting these values into equation 20 gives a σ of:

$$\sigma = \frac{0.013155}{\sqrt{\frac{1}{252}}} = 0.208832 = 20.88\%$$

This volatility means that the price of Diageo plc shares will fluctuate annually by 20.88%.

Evaluation of Drift μ

To evaluate μ the expected annual rate of return (drift) is: (Dmouj, 2006)

$$\mu = \frac{\hat{u}}{\tau} + \frac{1}{2}\sigma^2 \quad (21)$$

Inputting the values obtained above into equation 21 gives:

$$\mu = \frac{-0.00169}{\frac{1}{252}} + \frac{1}{2}0.208832^2 = -0.40411 = -40.41\%$$

This means that the expected share price will decrease annually by 40.41%.

(iii) Do these values of μ and σ make sense?
The value for μ for Diageo is -40.41% using data from 1 August 2019 to 31 October 2019 indicating a considerable annual decrease is share price. While Diageo share prices have continued to fall past the 31 October with a percentage change of 13.47% from 1 August 2019 to 13 December 2019 (MorningStar) it appears that this continued fall in share price supports the value for μ of -40.41%. Furthermore, observing figure 2 it appears that there is downward trend. However, given that this is a particularly significant shift while analysts (Ballard) seem to continue to rate this as a stable stock then the magnitude of this value is perhaps overestimated. The model may have failed to factor in a number of other variables, one particular factor

could be the dividends that where paid out on 8 August 2019 (MorningStar), while the model here does not assume any dividends are paid out.(Marathe and Ryan, 2005a).

The volatility σ is 20.88% which indicates how much the share prices will vary annually. A large part of this fluctuation in share price could be due to Brexit and the volatility in the Sterling against other global currencies. The geo-political uncertainty appears to have significant impact on Diageo as it was recently reported in the Financial Times (Rovnick) that Diageo has sold off a significant number of its emerging market currencies. Moreover Diageo themselves have stated in the 2019 annual report that Scotch whisky made up 25% of net sales (Menzes, 2019) illustrating just how closely the company share price is linked to ongoing Brexit negotiations and international trade prospects with the UK.

(iv) Estimating the expected value of Diageo on 30 November 2019 and comparing it with the published data.

The expected value $E(S_t)$ of a share price on a given time t can be estimated by(Dmouj, 2006):

$$E(S_t) = S_0 exp\left((\mu + \frac{\sigma^2}{2})t\right) \tag{22}$$

Where S_0 is the initial share price and t is the future time (in days).

Using 1 August 2019 as S_0 the expected value of Diageo shares on 30 November 2019 can be estimated (to the nearest 50p) by inputting the relevant value into 22 as follows:

$$E(S_{86}) = 3526exp\left((0.40511 + \frac{0.20832^2}{2})\frac{86}{252}\right) = £3094.50$$

The actual close on the 29 November 2019 (30 November was not a trading day) was $3165.00 (MorningStar) only 2.2% higher than the estimated value. It is also worth noting that only three trading days later the share price for Diageo matched this estimated value of $3094.50. If the parameter S_0 is changed to use the share price on the 31 October 2019 of $3164 and inputting into equation 22 it now gives $3052.50 a difference of 1.4% from the actual value. Given that as stated in the assumptions of Geometric Brownian Motion that share prices follow a Markov chain process this supports that only the current value of the share is relevant when predicting future prices.

Although there is a minor discrepancy between the predicted share price and actual share price this can be expected as it is modelling a real life problem and it is expected to have a small degree of error within such a model. Consequently, the model may still be considered as a good estimator of the stock price.

4 Conclusion

4.1 (i) Restrictions to the Model

A possible restriction to the model is that μ and σ are treated as constants and not as functions of time that fluctuate when trying to estimate share prices using these coefficients. Another restriction is that the equation 22 does not factor in W_t which accounts for the random nature at which share prices fluctuate. Lastly, the model does not factor in for dividend payouts this model could be adjusted to account for this (Merton et al., 1973).

4.2 (ii) Model Evaluation

Although Diageo's share prices have been falling in the 2019 annual report (Menzes, 2019), Diageo have addressed this by pointing out that this is largely due to global market instability and geo-political uncertainty. However, in the last decade Diageo share prices have performed roughly in-line with the broad market (MorningStar). To drive returns Diageo have taken an aggressive acquisition tactic(Ballard) and have persistently purchased competition in an attempt to retain and even grow their market share.

Conversely, it is difficult to truly model and predict the price of a stock, meaning even a strong profitable company will often randomly shift in price contrary to what many investors and analysts may expect. It is here where Geometric Brownian Motion attempts to take into account this random shift in share price. As seen above the predicted price for the 30 November 2019 was indeed close to the actual price.

However, although this model does take into account the random nature of stocks there are still multiple factors that will influence a share price such as: the exchange rate of different currencies, anticipated trade deals, taxation (especially as the alcohol industry is heavily taxed) and other such external factors that are not taken into account in this model.

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